PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

A NEW PROOF FOR ROCKAFELLAR'S CHARACTERIZATION OF MAXIMAL MONOTONE OPERATORS

S. SIMONS AND C. ZĂLINESCU

(Communicated by J. M. Borwein)

ABSTRACT. We provide a new and short proof for Rockafellar's characterization of maximal monotone operators in reflexive Banach spaces based on S. Fitzpatrick's function and a technique used by R. S. Burachik and B. F. Svaiter for proving their result on the representation of a maximal monotone operator by convex functions.

1. The result

Throughout this note $(X, \|\cdot\|)$ is a reflexive Banach space and X^* is its topological dual space whose dual norm is denoted by $\|\cdot\|_*$. Then the topological dual of $X \times X^*$ is $X^* \times X$, the pairing being given by $\langle (x, x^*), (u^*, u)\rangle := \langle x, u^*\rangle + \langle u, x^*\rangle$, where, as usual, $\langle x, u^*\rangle := u^*(x)$ for $x \in X$ and $u^* \in X^*$. Let $A : X \rightrightarrows X^*$ be a multivalued operator (or multifunction) whose graph $\operatorname{gr} A := \{(x, x^*) \mid x^* \in A(x)\}$ is nonempty. Recall that A is *monotone* if

$$\langle x - y, x^* - y^*\rangle \geq 0 \quad \forall\, (x, x^*),\ (y, y^*) \in \operatorname{gr} A;$$

A is *maximal monotone* if A is monotone and for any monotone operator $B : X \rightrightarrows X^*$ with $\operatorname{gr} A \subset \operatorname{gr} B$ we have that $A = B$.

It is well known (and obvious) that if $A : X \rightrightarrows X^*$ is a maximal monotone operator and $(y, y^*) \in X \times X^*$ then

$$\inf_{(a,a^*)\in \operatorname{gr} A} \langle a - y, a^* - y^*\rangle \leq 0, \text{ with equality } \Leftrightarrow (y, y^*) \in \operatorname{gr} A. \tag{1.1}$$

Define

$$g : X \times X^* \to \overline{\mathbb{R}}, \quad g(y, y^*) := \sup_{(a,a^*)\in \operatorname{gr} A} [\langle a, y^*\rangle + \langle y, a^*\rangle - \langle a, a^*\rangle]. \tag{1.2}$$

Then (1.1) can be written as

$$\forall\, (y, y^*) \in X \times X^*,\ \langle y, y^*\rangle \leq g(y, y^*), \text{ with equality } \Leftrightarrow (y, y^*) \in \operatorname{gr} A. \tag{1.3}$$

As Fitzpatrick [2] observed, g is a lower semicontinuous proper convex function when A is maximal monotone. In the next statement the subdifferential of g is meant in the sense of convex analysis; in particular the subdifferential is empty at a point outside the domain of the function.

2000 *Mathematics Subject Classification.* Primary 47H05; Secondary 26B25.
Key words and phrases. Maximal monotone operator, convex function, duality mapping.

Lemma 1.1. *Assume that A is maximal monotone and $(u^*, u) \in \partial g(v, v^*)$. Then $\langle v - u, v^* - u^* \rangle \le 0$. Moreover, if $\langle v - u, v^* - u^* \rangle = 0$ then $(u, u^*) \in \operatorname{gr} A$.*

Proof. From (1.3) we have that

$$\begin{aligned}\langle v - u, v^* - u^* \rangle &= \langle v, v^* \rangle - \langle v, u^* \rangle - \langle u, v^* \rangle + \langle u, u^* \rangle \\ &\le g(v, v^*) - \langle v, u^* \rangle - \langle u, v^* \rangle + \langle u, u^* \rangle .\end{aligned}$$

Fix $(a, a^*) \in \operatorname{gr} A$. Since $(u^*, u) \in \partial g(v, v^*)$ and using (1.3) again, we have that

$$g(v, v^*) \le g(a, a^*) - \langle (a, a^*) - (v, v^*), (u^*, u) \rangle = \langle a, a^* \rangle + \langle v - a, u^* \rangle + \langle u, v^* - a^* \rangle .$$

Therefore,

$$\langle v - u, v^* - u^* \rangle \le \langle a, a^* \rangle - \langle a, u^* \rangle - \langle u, a^* \rangle + \langle u, u^* \rangle = \langle a - u, a^* - u^* \rangle .$$

Hence

$$\langle v - u, v^* - u^* \rangle \le \inf_{(a,a^*) \in \operatorname{gr} A} \langle a - u, a^* - u^* \rangle . \tag{1.4}$$

The first part of the result follows from (1.1). If $\langle v - u, v^* - u^* \rangle = 0$, then (1.4) gives $\inf_{(a,a^*) \in \operatorname{gr} A} \langle a - u, a^* - u^* \rangle \ge 0$, and so, using again (1.1), we obtain that $(u, u^*) \in \operatorname{gr} A$. □

Recall that the duality mapping of X is the multifunction $J_X : X \rightrightarrows X^*$ defined by $J_X(x) := \{x^* \in X^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|_*^2\} = \partial\big(\tfrac{1}{2} \|\cdot\|^2\big)$; then $J_X^{-1}(x^*) = \partial\big(\tfrac{1}{2} \|\cdot\|_*^2\big)$. Using Lemma 1.1, we get the generalization in [7, Thm. 10.7] of Rockafellar's celebrated characterization of maximal monotone operators.

Theorem 1.2. *Let X be a reflexive Banach space and $A : X \rightrightarrows X^*$ be monotone. Then A is maximal monotone if and only if $\operatorname{gr} A + \operatorname{gr}(-J_X) = X \times X^*$.*

Proof. The proof of the sufficiency does not suppose the reflexivity of the space and is well known. Indeed, let $(y, y^*) \in X \times X^*$ be such that $\langle a - y, a^* - y^* \rangle \ge 0$ for every $(a, a^*) \in \operatorname{gr} A$. By hypothesis there exists $(a, a^*) \in \operatorname{gr} A$ and $(u, u^*) \in \operatorname{gr}(-J_X)$ such that $(y, y^*) = (a, a^*) + (u, u^*)$. Then

$$0 \le \langle a - y, a^* - y^* \rangle = \langle -u, -u^* \rangle = -\langle u, -u^* \rangle = -\|u\|^2 = -\|-u^*\|_*^2 ,$$

and so $u = 0$ and $u^* = 0$ (we have used the fact that $-u^* \in J_X(u)$). It follows that $(y, y^*) = (a, a^*) \in \operatorname{gr} A$. Hence A is maximal monotone.

Assume that A is maximal monotone. Let us prove that $(0, 0) \in \operatorname{gr} A + \operatorname{gr}(-J_X)$.

Consider g defined in (1.2); as observed above, g is a proper lower semicontinuous convex function. Consider the function $h : X \times X^* \to \overline{\mathbb{R}}$ defined by

$$h(x, x^*) := \tfrac{1}{2} \|x\|^2 + \tfrac{1}{2} \|x^*\|_*^2 + g(x, x^*).$$

It is obvious that h is proper, lower semicontinuous, (strongly) coercive and convex. The space $X \times X^*$ being reflexive, there exists (v, v^*) minimizing h on $X \times X^*$. Hence $(0, 0) \in \partial h(v, v^*)$. Moreover, because h is the sum of three convex functions, two of them being continuous, one obtains that

$$(0, 0) \in J_X(v) \times J_X^{-1}(v^*) + \partial g(v, v^*),$$

and so there exists $(u^*, u) \in \partial g(v, v^*)$ such that $-u^* \in J_X(v)$ and $-u \in J_X^{-1}(v^*)$ (equivalently $-v^* \in J_X(u)$). Using Lemma 1.1 we obtain that $\langle v - u, v^* - u^* \rangle \le 0$.

Because $-u^* \in J_X(v)$ and $-v^* \in J_X(u)$ we have that $\langle v, -u^*\rangle = \|v\|^2 = \|u^*\|_*^2$, $\langle u, -v^*\rangle = \|u\|^2 = \|v^*\|_*^2$, and so

$$(1.5) \qquad \begin{aligned} 0 &\geq \langle v-u, v^*-u^*\rangle = \langle v, v^*\rangle + \langle v, -u^*\rangle + \langle u, -v^*\rangle + \langle u, u^*\rangle \\ &\geq \|v\|^2 - 2\|v\|\cdot\|u\| + \|u\|^2 = (\|v\| - \|u\|)^2 \geq 0. \end{aligned}$$

Hence each inequality in (1.5) is in fact an equality. Therefore $\langle v-u, v^*-u^*\rangle = 0$ and $\|v\| = \|u\|$ $(= \|v^*\|_* = \|u^*\|_*)$, $\langle -u, u^*\rangle = \|v\|\cdot\|u\| = \|-u\|^2 = \|u^*\|_*^2$; thus $u^* \in J_X(-u)$. Using the last part of Lemma 1.1 we obtain that $(u, u^*) \in \operatorname{gr} A$. Because $(-u, -u^*) \in \operatorname{gr}(-J_X)$, we deduce that $(0,0) = (u,u^*) + (-u,-u^*) \in \operatorname{gr} A + \operatorname{gr}(-J_X)$.

Let $(y, y^*) \in X \times X^*$ be fixed. Taking $A' : X \rightrightarrows X^*$ with $\operatorname{gr} A' = \operatorname{gr} A - (y, y^*)$, A' is maximal monotone. By what precedes we obtain that $(0,0) \in \operatorname{gr} A' + \operatorname{gr}(-J_X)$, that is $(y, y^*) \in \operatorname{gr} A + \operatorname{gr}(-J_X)$. The conclusion follows. □

The proof of this theorem in [7, Thm. 10.6] is based on several results, some of them using minimax theorems (though it is now known that the minimax theorems can be replaced by an appropriately generalized form of the Hahn–Banach theorem). When J_X and J_X^{-1} are single valued the preceding result yields (easily) Rockafellar characterization of maximal monotonicity of A in [6] (see also [7, Thm. 10.7, Rem. 10.8]):

Theorem 1.3. *Let X be a reflexive Banach space and $A : X \rightrightarrows X^*$ be monotone. If A is maximal monotone then $A + J_X$ is onto. Conversely, if $A + J_X$ is onto and J_X, J_X^{-1} are single valued then A is maximal monotone.*

Without asking J_X and J_X^{-1} be single-valued in the last part of the preceding theorem the statement is not true as can be seen from [7, page 39].

Our proof of Theorem 1.2 is inspired by the proof of [1, Thm. 3.1], where a similar function to h (above) is considered. In order to prove their result, Burachik and Svaiter renorm the space X and apply Rockafellar's theorem; applying Theorem 1.2 there is no need to renorm the space.

It is possible to prove Theorem 1.2 using a characterization of maximal monotone subsets of $X \times X^*$ given by Martinez-Legaz and Théra in [4] and some properties of convex functions associated to monotone subsets from Penot [5], but the use of Lemma 1.1 leads to a much shorter proof.

Without asking X to be reflexive the statement of Theorem 1.2 does not hold. Indeed, taking $A : X \rightrightarrows X^*$ defined by $A(x) := \{0\}$, A is maximal monotone and $\operatorname{gr} A + \operatorname{gr}(-J_X) = X \times \operatorname{Im} J_X$. But, for X a Banach space, J_X is onto if and only if X is reflexive.

If X is a nonreflexive Banach space, we write $\widehat{\ }$ for the canonical map from X into its bidual X^{**}. Now let $A\colon X \rightrightarrows X^*$ be maximal monotone. We say that A is *of type (NI)* if, whenever $(y^{**}, y^*) \in X^{**} \times X^*$,

$$\inf_{(a,a^*)\in \operatorname{gr} A} \langle a^* - y^*, \widehat{a} - y^{**}\rangle \leq 0.$$

Furthermore, in [3], Gossez introduced the multifunction $\overline{A}\colon X^{**} \rightrightarrows X^*$ defined by:

$$y^* \in \overline{A} y^{**} \iff \inf_{(a,a^*)\in \operatorname{gr} A} \langle a^* - y^*, \widehat{a} - y^{**}\rangle \geq 0.$$

The following result was proved in [7], Lemma 27.5, and was used in [7] to give a precise description of the closure of the range of a maximal monotone operator of

Gossez's "type (D)", and in [8] to prove that certain maximal monotone multifunctions have an approximate "Brøndsted–Rockafellar" property. *Let* $A\colon X \rightrightarrows X^*$ *be maximal monotone of type (NI). Then there exist* $u^{**} \in X^{**}$ *and* $u^* \in \overline{A}u^{**}$ *such that* $u^{**} \in -J_{X^*}(u^*)$. It would be very nice if the techniques of this paper can be used to prove this result, but we do not know if it is the case.

References

[1] R. S. Burachik and B. F. Svaiter, *Maximal monotonicity, conjugation and the duality product*, Tech. report, IMPA, Rio de Janeiro, 2002.

[2] S. Fitzpatrick, *Representing monotone operators by convex functions*, Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988), Austral. Nat. Univ., Canberra, 1988, pp. 59–65.

[3] J.-P. Gossez, *Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs* J. Math. Anal. Appl. **34** (1971), 371–395.

[4] J. E. Martinez-Legaz and M. Théra, *A convex representation of maximal monotone operators*, J. Nonlinear Convex Anal. **2** (2001), 243–247.

[5] J.-P. Penot, *The relevance of convex analysis for the study of monotonicity*, Tech. report, University of Pau, Pau, 2002.

[6] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[7] S. Simons, *Minimax and monotonicity*, Springer-Verlag, Berlin, 1998.

[8] S. Simons, *Maximal monotone multifunctions of Brøndsted–Rockafellar type*, Set–Valued Anal. **7** (1999), 255–294.

Department of Mathematics, University of California, Santa Barbara, CA 93106, U.S.A.

E-mail address: simons@math.ucsb.edu

University "Al. I. Cuza" Iaşi, Faculty of Mathematics, Bd. Copou Nr. 11, 6600 Iaşi, Romania

E-mail address: zalinesc@uaic.ro